Available online at www.sciencedirect.com
ScienceDirect
Discrete Applied Mathematics 156 (2008) 2636–2646
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

Stability of two player game structures

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Received 24 April 2006; received in revised form 22 August 2007; accepted 7 November 2007
Available online 26 December 2007

Abstract

We have extended a two player game-theoretical model proposed by V. Gurvich [To theory of multi-step games, USSR Comput. Math and Math. Phys. 13 (1973)] and H. Moulin [The Strategy of Social Choice, North Holland, Amsterdam, 1983]: All the considered *game situations* are framed by the same *game structure*. The structure determines the families of *potential decisions* of the two players, as well as the subsets of *possible outcomes* allowed by pairs of such choices. To be a *solution of a game*, a pair of decisions has to determine a *(pure) functional equilibrium* of the *situational* pair of *payoff mappings* which transforms the *realized outcome* into *real-valued rewards* of the players. Accordingly we understand that a structure is *stable*, if it admits functional equilibria for all possible game situations; and that it is *complete*, if every situation that only *partitions the potential outcomes*, is *dominated* by one of the players. We have generalized and strengthened a theorem by V. Gurvich [Equilibrium in pure strategies, Soviet Math. Dokl. 38 (1989)], proving that a *proper* structure is *stable* iff it is *complete*. Additional results provide game-theoretical insight that focuses the inquiry on the *complexity* of the *stability decision problem*; in particular, for *coherent* structures.

These results also have combinatorial importance because every structure is characterized by a pair of *hypergraphs* [C. Berge, Graphes et Hypergraphes, Dunod, 1970] over a common *ground set*. The structure is *dual* (*complete*/*coherent*) iff the *clutter* of one hypergraph equals (includes/is included in) the *blocker* of the other one. So, for non-void coherent structures, the *stability decision problem* is equivalent to the much studied subexponential [M.L. Fredman, L. Khachiyan, On the complexity of dualization of monotone disjunctive normal forms, J. Algorithms 21 (1996)] *hypergraph duality decision problem*.

Keywords: Game form; Equilibrium; Stable structure; Hypergraph transversal; Duality problem; Polynomial reduction

1. Introduction

Since its beginning [3] game theory has had difficulties in defining what should be considered a *solution of a game*. Because even for the most appealing formalizations, there are *games* that have *no solution*; see, for instance, [12]. Such *unsolvable games* would typically frustrate involved players who could not gasp the tempting rewards. Therefore they also worried game theoreticians. This is probably the reason why some authors began to qualify the original game-theoretical convictions, and started to visualize *structures* that coin or delimit the *universe of situational games* that may take place, as well as what should be considered their *solutions*. Then it became natural to *blame* the prevailing

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doi:10.1016/j.dam.2007.11.009

structures for the *frustrations* they were not able to prevent. It became sensible to ask, what *particular properties* such a *structure* must have, so that *all possible games* permit solutions. According to some authors [13,9] only such a *structure* may considered to be *stable*; otherwise, *unsettled game situations*, through the *frustrations* they produce, would react against their *structural determinants* and *destabilize* them.

For this paper's purpose we adopt a *categorical formal version* of this point of view: We assume that a structure essentially is a *solving principle* put into practice; and that it therefore gets questioned by any *potential failure of the principle*. *Universal principles* are like *mathematical conjectures*: they get *disproved by counterexamples*. So any potential situation that resists solution, questions the structure; even if the situation does not really take place. Only *universal solvability* guarantees *stability* of the structure.

In this paper we consider such a *structural stability problem*. We are going to build on a game-theoretical model that adopts the classical *Nash equilibrium (in pure strategies)* solution concept; and assumes that the games that can take place in various situations, are all framed by a given *game form* [7,8,2,10]. A game form is called *Nash-solvable* if all the games that the form allows have Nash equilibria. A peak result of this inquiry – Theorem 1 of [10] – then proves that if the game form, structurally, only includes two players, then the game form will be Nash-solvable iff the game form is *tight*; i.e., iff *the pair of hypergraphs that the form determines* – one for each player – is *dual*. We extend this game-theoretic model from game forms to *structures* that are *pairs of hypergraphs over a common finite ground set*. So this class of structures includes, as special cases, those *determined by game forms* [10]. Since now all such pairs of hypergraphs are possible structures, this model also allows us to express game-theoretical reformulation of all the questions raised by the *hypergraph duality theory* [6,11,14,4].

The paper is organized as follows: In Section 2 we first gather some *classical results on hypergraphs* which we shall later use to develop our game-theoretical considerations. In Section 3 we introduce to the indispensable *game-theoretical background*. In Section 4 we recycle *classical blocker theory* [5] to analyze the *solvability of structures* that are exposed to *antagonistic games*. In Section 5 we define *structural stability* and derive the new *minimal transversal* considerations that sustain our main results. Finally, in Section 6, we prove that one can restrict the attention to *proper game structures*, or even to game forms.

2. Hypergraphs

In all what follows, A will be a given finite *ground set*.

Definition. A *hypergraph* on the ground set A, is a family $\mathcal{H}$ of subsets of A. $\mathcal{H}$ is called *proper*, if $\mathcal{H} \neq \emptyset$ and $\emptyset \notin \mathcal{H}$. The *domain* of $\mathcal{H}$ is $\bigcup\{X \in \mathcal{H}\} \subseteq A$. If it equals A, then $\mathcal{H}$ has *full domain*. The original definition [1] required of hypergraphs to be proper and have full domain. Although such proper hypergraphs are the ones that will most interest us, like other authors [4] we will not restrict the notion. The *size* of $\mathcal{H}$ is $\kappa(\mathcal{H}) := |A| \cdot |\mathcal{H}| \in \mathbb{N}$.

Definition. Given a hypergraph $\mathcal{H}$, let $\nu(\mathcal{H}) := \{W \subseteq A; \exists X \in \mathcal{H}, X \subseteq W\}$ denote the family of subsets of A that are *responded* [14] by (members of) $\mathcal{H}$ — or, according to other authors, the *clutter* of $\mathcal{H}$. Note that $\nu \circ \nu = \nu$.

Definition. A *(hypergraph) structure on A* is a cartesian product $\mathcal{G} := \mathcal{H} \times \mathcal{K}$ of two hypergraphs over the same *ground set* A. $\mathcal{G}$ is called *proper*, if $\mathcal{H}$ and $\mathcal{K}$ are proper. $\mathcal{G}$ has *unique domain* [6], if $\bigcup\{X \in \mathcal{H}\} = \bigcup\{Y \in \mathcal{K}\}$. The *size* of the structure $\mathcal{G}$ is $\kappa(\mathcal{G}) := |A| \cdot (|\mathcal{H}| + |\mathcal{K}|) \in \mathbb{N}$.
The structure $\mathcal{G}$ is called *coherent* [14], if for all partitions (W, Z) of A, either $W \notin \nu(\mathcal{H})$ or $Z \notin \nu(\mathcal{K})$.
This property can be decided in *polynomial time* – i.e., with a computational effort that can be bounded by a polynomial in $\kappa(\mathcal{G})$ – since, as is easy to see, it holds iff $\forall (X, Y) \in \mathcal{H} \times \mathcal{K}, X \cap Y \neq \emptyset$.
The structure $\mathcal{G}$ is called *complete* [14], if for all partitions (W, Z) of A, either $W \in \nu(\mathcal{H})$ or $Z \in \nu(\mathcal{K})$. The corresponding *completeness decision problem* is *coNP-complete* [14].
The structure $\mathcal{G}$ is called *dual*, if it is coherent and complete. The corresponding *duality decision problem* can be solved in *subexponential time* [6].
Note that each of the three considered *structural properties* – like all that will interest us in what follows – is *symmetric*: it holds for $\mathcal{H} \times \mathcal{K}$ iff it holds for $\mathcal{K} \times \mathcal{H}$.

Definition. Given a hypergraph $\mathcal{H}$, let $\tau(\mathcal{H}) := \{Z \subseteq A; \forall X \in \mathcal{H}, X \cap Z \neq \emptyset\}$ denote the family of all *transversal* of $\mathcal{H}$ – or, according to other authors, the *blocker* of $\mathcal{H}$.

The *holistic form* of the operator τ – that does not depend on the particularities of the binary relation $X \cap Z \neq \emptyset$ – immediately implies that it is *antitone* – i.e., if $\mathcal{H} \subseteq \mathcal{H}'$, then $\tau(\mathcal{H}) \supseteq \tau(\mathcal{H}')$. Moreover, it implies that $\tau \circ \tau$ is *extensive* – i.e., $(\tau \circ \tau)(\mathcal{H}) \supseteq \mathcal{H}$; so $\tau^2 \supseteq \iota$, where $\tau^2 := \tau \circ \tau$ and ι is the identity operator. These *hologrammatic properties* imply that τ is a *bi-potent operator* – i.e., $\tau^3 = \tau$; since extensiveness of τ^2 implies $\tau^3 = \tau^2 \circ \tau \supseteq \iota \circ \tau = \tau$; and antitonicity of τ yields $\tau^3 = \tau \circ \tau^2 \subseteq \tau \circ \iota = \tau$.

Moreover, the particular properties of the relation that defines τ, imply a simple *Theorem of Alternatives* [5]: for all partitions (W, Z) of A, either $W \in \nu(\mathcal{H})$ or (exclusive) $Z \in \tau(\mathcal{H})$ – because $W \in \nu(\mathcal{H})$ iff $\exists X \in \mathcal{H}$ with $X \cap Z = \emptyset$; i.e., iff $Z \notin \tau(\mathcal{H})$. So, since evidently $\nu \circ \tau = \tau$, one has $W \in \nu(\mathcal{H})$ iff $Z \notin (\nu \circ \tau)(\mathcal{H})$; i.e., iff $W \in (\tau \circ \tau)(\mathcal{H})$. So $\tau \circ \tau = \nu$; that is, τ is the *square root* of ν.

Corollary of Alternatives. Given a structure $\mathcal{G} := \mathcal{H} \times \mathcal{K}$, each of the assertions of the following three lines is equivalent to each one of the same line:

$\mathcal{G}$ is coherent.	For all partitions (W, Z) of A either $W \in \tau(\mathcal{K})$ or $Z \in \tau(\mathcal{H})$.	$\tau(\mathcal{H}) \supseteq \nu(\mathcal{K})$	$\nu(\mathcal{H}) \subseteq \tau(\mathcal{K})$
$\mathcal{G}$ is complete.	For all partitions (W, Z) of A either $W \notin \tau(\mathcal{K})$ or $Z \notin \tau(\mathcal{H})$.	$\tau(\mathcal{H}) \subseteq \nu(\mathcal{K})$	$\nu(\mathcal{H}) \supseteq \tau(\mathcal{K})$
$\mathcal{G}$ is dual.	For all partitions (W, Z) of A: $W \notin \tau(\mathcal{K})$ iff $Z \in \tau(\mathcal{H})$.	$\tau(\mathcal{H}) = \nu(\mathcal{K})$	$\nu(\mathcal{H}) = \tau(\mathcal{K})$

Definition. Given a structure $\mathcal{G} := \mathcal{H} \times \mathcal{K}$, let $\nu(\mathcal{G}) := \nu(\mathcal{H}) \times \nu(\mathcal{K})$ denote the *response structure of* $\mathcal{G}$, and $\tau(\mathcal{G}) := \tau(\mathcal{K}) \times \tau(\mathcal{H})$ denote the *transversal structure of* $\mathcal{G}$. Note that $\nu(\mathcal{G}) = \tau^2(\mathcal{G})$ and $\tau(\mathcal{G}) = \tau^3(\mathcal{G})$.
So each of the assertions of the following three lines is equivalent to each one of the same line:

$\mathcal{G}$ is coherent.	$\mathcal{G} \subseteq \tau(\mathcal{G})$	$\nu(\mathcal{G})$ is coherent.	$\nu(\mathcal{G}) \subseteq \tau(\mathcal{G})$	$\tau(\mathcal{G})$ is complete.
$\mathcal{G}$ is complete.		$\nu(\mathcal{G})$ is complete.	$\nu(\mathcal{G}) \supseteq \tau(\mathcal{G})$	$\tau(\mathcal{G})$ is coherent.
$\mathcal{G}$ is dual.		$\nu(\mathcal{G})$ is dual.	$\nu(\mathcal{G}) = \tau(\mathcal{G})$	$\tau(\mathcal{G})$ is dual.

Definition. Given a structure $\mathcal{G} := \mathcal{H} \times \mathcal{K}$, let $\mu(\mathcal{H}) := \{X \in \mathcal{H}; \forall X' \in \mathcal{H}, X' \not\subset X\}$ denote the family of *minimal members* of $\mathcal{H}$; and let $\mu(\mathcal{G}) := \mu(\mathcal{H}) \times \mu(\mathcal{K})$ denote the *minimal kernel* of $\mathcal{G}$.
It is a *kernel*, because $\mu(\mathcal{G}) \subseteq \mathcal{G}$; and because if $\mathcal{G}' \subseteq \mathcal{G}$, then $\mathcal{G}' \setminus \mu(\mathcal{G}') \subseteq \mathcal{G} \setminus \mu(\mathcal{G})$; so $(\mu \circ \mu)(\mathcal{G}) = \mu(\mathcal{G})$.
Also note that $(\nu \circ \mu)(\mathcal{G}) = \nu(\mathcal{G})$ and $(\tau \circ \mu)(\mathcal{G}) = \tau(\mathcal{G})$; and that therefore $\mathcal{G}$ is coherent (complete, dual) iff $\mu(\mathcal{G})$ is coherent (complete, dual).
Finally note that $\mu(\mathcal{G})$ – but not necessarily $\nu(\mathcal{G})$ nor $\tau(\mathcal{G})$ – can be determined in polynomial time; since, to determine $\mu(\mathcal{H})$, it suffices to *traverse* $\mathcal{H}$ and *discard* the $X \in \mathcal{H}$ for which $\exists X' \in \mathcal{H}$ with $X' \subset X$.

So the considered structural properties of $\mathcal{G}$ are shared by its minimal kernel $\mu(\mathcal{G})$. Therefore some authors [4] save the denotation *dual pairs of hypergraphs* for dual and minimal structures. We shall not do so, but later – Section 6, Lemma 4 – we will show that one could as well consider only *minimal and critical substructures*.

Definition. Let $\sigma(\mathcal{H}) := \{Z \subseteq A; \forall a \in Z, \exists X \in \mathcal{H}$ with $X \cap Z = \{a\}\}$ denote the family of subsets that are *critical, subject to* $\mathcal{H}$; and let $\sigma(\mathcal{G}) := \sigma(\mathcal{K}) \times \sigma(\mathcal{H})$.
Note that σ is a monotone operator; and $\forall Z \in \sigma(\mathcal{H})$, if $Y \subseteq Z$, then $Y \in \sigma(\mathcal{H})$. So $\nu(\mathcal{G}) \cap \sigma(\mathcal{G}) \subseteq \nu(\mu(\mathcal{G}) \cap \sigma(\mathcal{G}))$.
$\mathcal{G}$ is called *critical*, if $G \subseteq \sigma(\mathcal{G})$. This property is polynomially decidable.

Definition. Let $\lambda(\mathcal{H}) := \sigma(\mathcal{H}) \cap \tau(\mathcal{H})$ denote the family of *slices* of $\mathcal{H}$; and let $\lambda(\mathcal{G}) := \lambda(\mathcal{K}) \times \lambda(\mathcal{H})$ denote the *slice structure* of $\mathcal{G}$.

Note that $(\mu \circ \tau)(\mathcal{G}) = \lambda(\mathcal{G})$. Using the already known operator equations and just replacing equal terms, this implies $(\lambda \circ \nu)(\mathcal{G}) = \lambda(\mathcal{G})$ and $(\lambda \circ \tau)(\mathcal{G}) = \mu(\mathcal{G})$, $(\nu \circ \lambda)(\mathcal{G}) = \tau(\mathcal{G})$ and $(\tau \circ \lambda)(\mathcal{G}) = \nu(\mathcal{G})$.
So λ is a *bi-potent operator* – i.e., $\lambda^3 = \lambda$ – that is the *square root* of μ – i.e., $\lambda^2 = \mu$.
So $\mathcal{G}$ is coherent iff $\mu(\mathcal{G}) \subseteq \tau(\mathcal{G})$. $\mathcal{G}$ is complete iff $\nu(\mathcal{G}) \supseteq \lambda(\mathcal{G})$. $\mathcal{G}$ is dual iff $\mu(\mathcal{G}) = \lambda(\mathcal{G})$. This last condition is equivalent to each one of the following: $\lambda(\mathcal{H}) = \mu(\mathcal{K})$; $\mu(\mathcal{H}) = \lambda(\mathcal{K})$.
So $\mu(\mathcal{G})$ is coherent (complete, dual) iff $\lambda(\mathcal{G})$ is complete (coherent, dual).

3. Games

Let us now briefly introduce the game-theoretical background of the main mathematical results in this paper. Following [7,8,2,10], we define the concept of a game form:

Definition. Given a finite set A of *potential outcomes*, a *(two players, functional) game form* (H, g, K) on A, is a pair (H, K) of finite sets of *potential decisions* – one for each of the two *players* associated to H, K, respectively – and a *decision function* $g : H \times K \to A$. So a game form is naturally represented by an $H \times K$ matrix g with entries from A.

For instance we could have $A := \{a, b, c, d\}$, $H := \{x_1, x_2\}$, $K := \{y_1, y_2, y_3\}$ and a function g specified by:

g	y_1	y_2	y_3
x_1	a	b	c
x_2	c	d	d

Definition. Let F be the family of functions $z : A \to \mathbb{R}$. A *game* for a given form (H, g, K), is (a situation) specified by a pair $(w, z) \in F \times F$ of *payoff mappings*, that for any outcome $a \in A$, assigns $(w(a), z(a))$ to the players (associated to) (H, K); and thus, for a pair of decisions $(x, y) \in H \times K$, pays $(w(g(x, y)), z(g(x, y)))$ to the corresponding players.

Definition. Given $B \subseteq A$ and $w : A \to \mathbb{R}$, let $\bigvee(w, B) := \{a \in B; \forall b \in B, w(a) \geq w(b)\}$ $(\bigwedge(w, B) := \{a \in B; \forall b \in B, w(a) \leq w(b)\})$ denote the subset of, *according to* w, *maximal (minimal) elements* of B.

Definition. A pair of decisions $(x, y) \in H \times K$ is a *Nash equilibrium* of such a game (w, z), if $g(x, y) \in \bigvee(w, \{g(x', y); x' \in H\}) \cap \bigvee(z, \{g(x, y'); y' \in K\})$; i.e., if $\forall x' \in H$, $w(g(x, y)) \geq w(g(x', y))$, and $\forall y' \in K$, $z(g(x, y)) \geq z(g(x, y'))$. In words: given such a pair of decisions (x, y), none of the two players is *motivated to change* his decision – to x', y', respectively – as long as the other one does not.

For instance, for the game form specified above, consider the situation:

A	a	b	c	d
w	$+2$	-2	0	-1
z	-1	$+3$	$+1$	$+2$

If $x = x_1$, then $g(x, y) \in \bigvee(z, \{g(x, y'); y' \in K\})$ requires $y = y_2$. But $g(x_1, y_2) = b$, $g(x_2, y_2) = d$ and $w(b) < w(d)$; so (x_1, y_2) is not an equilibrium. On the other hand, if $x = x_2$, then, since $z(c) < z(d)$, $y \in \{y_2, y_3\}$ is required. But only (x_2, y_2) is, in fact, a Nash equilibrium. If instead we modify the *situational incidence* of the potential outcome d, redefining $w(d) := z(d) := 0$, then a similar analysis shows that the specified game has no Nash equilibria.

Definition. A *correspondence game form* is a generalized game form that, instead of a *decision function* $g : H \times K \to A$, yields a *decision correspondence* $g : H \times K \to \mathcal{P}(A)$ – where $\mathcal{P}(A)$ is the family of all subsets of A. So, now the situational pair of decisions $(x, y) \in H \times K$ only *restricts* the potential situational outcomes to $g(x, y) \subseteq A$. When it *allows* more than one, it *leaves the final choice* $a \in g(x, y)$ *to chance*. We say that the correspondence game form is *proper*, if $H \neq \emptyset \neq K$ – so there exists at least one pair $(x, y) \in H \times K$ – and $\forall(x, y) \in H \times K$, $g(x, y) \neq \emptyset$ – so there exists at least one $a \in g(x, y)$.

One can also generalize the notion of *equilibria* for correspondence game forms. Implicitly we will do it in what follows: Beside game forms, generalizing a proposal of [10], we shall base our considerations on *structural models* of game-theoretical frames:

Definition. A *game structure* for the pair of players H, K, is a structure $\mathcal{G} := \mathcal{H} \times \mathcal{K}$ on A that, game theoretically, offers the families $\mathcal{H}$, $\mathcal{K}$, of potential choices for the respective players; and that when $(X, Y) \in \mathcal{G}$ is chosen, restricts the possible situational outcomes to $X \cap Y \subseteq A$ – possibly leaving the final choice to chance.

Given a game structure $\mathcal{G} := \mathcal{H} \times \mathcal{K}$ and a game situation $(w, z) \in F \times F$, a *(pure but weak) equilibrium* for (w, z) is a pair $(X, Y) \in \mathcal{G}$ such that $\bigvee(w, Y) \cap \bigvee(z, X) \neq \emptyset$; i.e., such that there exists an outcome $a \in X \cap Y$, that according to each of the two players, given the other player decision, is *maximal*.

A *strong equilibrium* is an equilibrium $(X, Y) \in \mathcal{G}$ such that $X \cap Y \subseteq \bigvee(w, Y) \cap \bigvee(z, X)$. We understand that only such strong equilibria *sufficiently settle a game situation*; because only then any possibly chosen $a \in X \cap Y$ will always be *maximal*.

But – in the absence of the predetermined decision functions of (functional) game forms – if *nothing should be left to chance*, we understand that game situations are to be *settled* by *functional equilibria*; i.e., equilibria $(X, Y) \in \mathcal{G}$ with $|X \cap Y| = 1$; because only then the outcome $a \in X \cap Y$ is *functionally determined* by the players decisions (X, Y).

Definition. Every (correspondence) game form (H, g, K) on A specifies *its game structure* $\mathcal{G} := \mathcal{H} \times \mathcal{K}$, where $\mathcal{H} := \{(\bigcup)\{g(x, y);\ y \in K\};\ x \in H\}$ and $\mathcal{K} := \{(\bigcup)\{g(x, y);\ x \in H\};\ y \in K\}$.

Note that the *equilibria* notions for structures yield sound *structural expressions* of the *Nash equilibria* of game forms; because if (H, g, K) is a (functional) game form and $\mathcal{G} := \mathcal{H} \times \mathcal{K}$ is its structure, then, although this structure retains only part of the game form features, it is able to *reproduce* the Nash equilibria of the game form: If $(X, Y) \in \mathcal{H} \times \mathcal{K}$ *represents* $(x, y) \in H \times K$, i.e., if $X = \{g(x, y');\ y' \in K\}$ and $Y = \{g(x', y);\ x' \in H\}$, then, given any $(w, z) \in F \times F$:

If (x, y) is a Nash equilibrium, then (X, Y) is an equilibrium – but not necessarily a strong one.

If (X, Y) is a strong equilibrium, then (x, y) is a Nash equilibrium – but the last does not necessarily hold if (X, Y) is *weak*.

For instance, the structure of the game form presented above, can be specified, exhibiting the characteristic functions x^1, x^2, y^1, y^2, y^3 of the subsets $X^1, X^2, Y^1, Y^2, Y^3 \subseteq A$ that define the structure $\mathcal{H} \times \mathcal{K}$:

A	a	b	c	d
x^1	1	1	1	0
x^2	0	0	1	1
y^1	1	0	1	0
y^2	0	1	0	1
y^3	0	0	1	1

Reconsider the (modified) situation that could not be Nash equilibrated by the game form of our example; or consider the following *qualitative version* of it:

A	a	b	c	d
w	1	0	0	0
z	0	1	1	0

Then (X^2, Y^3) yields a (weak) equilibrium, because $c \in \bigvee(w, Y^3) \cap \bigvee(z, X^2)$. Nonetheless, this situation allows no strong equilibria.

On the other hand every game structure $\mathcal{G}$ defines *its correspondence game form* $(\mathcal{H}, g, \mathcal{K})$, where the correspondence $g : \mathcal{G} \to \mathcal{P}(A)$ is specified in a standard way: $\forall (X, Y) \in \mathcal{H} \times \mathcal{K}$, $g(X, Y) := X \cap Y$. But note that game structures yield a slightly more general *game-theoretical frame* than the correspondence game forms: The structure $\bar{\mathcal{G}} := \bar{\mathcal{H}} \times \bar{\mathcal{K}}$ of the correspondence game form of a game structure $\mathcal{G}$, although it has the same correspondence game form, is not necessarily equal to the original $\mathcal{G}$: If the domains of $\mathcal{G}$ are $B := \bigcup\{X \in \mathcal{H}\}$ and $C := \bigcup\{Y \in \mathcal{K}\}$, respectively, then $\bar{\mathcal{H}} = \{X \cap C;\ X \in \mathcal{H}\}$ and $\bar{\mathcal{K}} = \{B \cap Y;\ Y \in \mathcal{K}\}$. So $\bar{\mathcal{G}}$, like all structures of game forms, has unique domain; in this case, equal to $B \cap C$. Therefore $\bar{\mathcal{G}} = \mathcal{G}$ iff $B = C$.

In Section 6 we will present game-theoretical reasons to focus on game structures with special properties, like domain uniqueness and coherence. Since some of these reasons are based on results that do not require these special properties, we will first present the general considerations.

4. Solvability

In this Section we shall present our first game-theoretical result: a structural generalization and strengthening of the following theorem of [7]: A (functional) game form (H, g, K) is *solvable* – i.e., every *zero-sum game* has a Nash equilibrium – iff its structure is complete.

Definition. A game situation is *antagonistic*, if its pair of payoff functions $(w, z) \in F \times F$ is such that each of these functions is an *antitone variation* of the other one; i.e., if $\forall (a, b) \in A \times A$, $w(a) < w(b)$ iff $z(a) > z(b)$.

Note that the much studied *constant-sum games*, where $\exists r \in \mathbb{R}$ such that $\forall a \in A$, $w(a) + z(a) = r$, are special antagonistic games; where any *improving* of one player's payoff has to be *paid* by the other one.

If, moreover, $r = 1$ and $\forall a \in A$, $w(a), z(a) \in \{0, 1\}$, then we say that (w, z) specifies a *competition (game)*. Any such competition is characterized by the partition (W, Z) of A – defined such that (w, z) presents the characteristic functions of (W, Z) – that specifies the subsets of *winning outcomes* for players H and K respectively. Player H *dominates* such a competition (w, z), if $W \in \nu(\mathcal{H})$ – i.e., if W is *responded* by $\mathcal{H}$, and thus player H can *ensure* that K does not *win* – and player K *dominates* the competition, if $Z \in \nu(\mathcal{K})$.

Lemma 1. *If a competition has an equilibrium, then (at least) one of the two players dominates the competition.*

Proof. Let (W, Z) be the partition of A determined by (w, z), let $(X, Y) \in \mathcal{H} \times \mathcal{K}$ be an equilibrium for (w, z), and let $a \in \bigvee(w, Y) \cap \bigvee(z, X)$. If $a \in Z$, then $w(a) = 0$; and therefore $\forall b \in Y$, $w(b) = 0$; so $Y \subseteq Z$. Accordingly, if $a \in Y$, then $X \subseteq W$. So, since either $a \in Z$ or $a \in W$, one of the two players will dominate the competition. □

This result is important for a *political version* of our game-theoretical model: Suppose that players H, K, are two *political parties*, and that the abstract *potential outcomes* gathered in A, now are human-like *managing agents*. Our model tells us that, if in a given situation $(w, z) \in F \times F$ the parties restrict the *representatives they would accept* to $X \in \mathcal{H}$ and $Y \in \mathcal{K}$ respectively, the situational choice – if one is possible – would have to distinguish an agent $a \in X \cap Y$. Then the *elected agent* would *manage* the situation, *paying back* $w(a)$, respectively $z(a)$, to the two parties that supported his appointment.

But in particular, if (w, z) is a competition and $W := \{a \in A; w(a) = 1\}$, $Z := \{a \in A; z(a) = 1\}$, then the *refunding* to the parties only depend on the *elected agent's situational position*: if $a \in W$, H will *win* and otherwise K will *succeed*. So it is natural to assume that W, Z, are the subsets of agents, that *share* the situational propositions of the parties H and K, respectively. Therefore, for such *elections*, whenever they can, the parties will only forward *subsets of candidates* $X \in \mathcal{H}$ and $Y \in \mathcal{K}$ such that $X \subseteq W$ and $Y \subseteq Z$, respectively; since otherwise they would be *supporting opponents*. The result stated in Lemma 1 guarantees, that the existing equilibria of competitions can in fact be *politically implemented*; that is, that at least one of the two parties will be able to *dominate the situational antagonism* (W, Z).

Lemma 2. *A proper structure $\mathcal{G} := \mathcal{H} \times \mathcal{K}$ is complete iff all its games $(w, z) \in F \times F$ allow equilibria.*

Proof. Lemma 1 guarantees that if all games allow equilibria, the considered structure is complete. Conversely, given a game (w, z), let $S := \bigcup\{\bigvee(w, Y); Y \in \mathcal{K}\}$ and $T := \bigcup\{\bigvee(z, X); X \in \mathcal{H}\}$. When $\mathcal{G}$ is proper and complete, $(S, T) \in \tau(\mathcal{G})$; and therefore – Section 2, Corollary of Alternatives – $S \cap T \neq \emptyset$, so (w, z) allows equilibria. □

The structure of our example of the last Section 3 is *incomplete*. We can prove it, considering the competition specified by the following payoff mappings:

A	a	b	c	d
w	1	0	0	1
z	0	1	1	0

This competition cannot be dominated by the players; although it differs only minimally from the last considered *qualitative situation* that did allow an equilibrium.

If, to *respond* to z, we add $y^4 := z$ to $\mathcal{K}$, the structure remains incomplete. An exhaustive analysis then proves that if one wants to *complete* it – with as few as possible additional coherent hyperedges – then one can add $X^3 := \{a, b, d\}$

to $\mathcal{H}$ and $Y^5 := \{a, d\}$ to $\mathcal{K}$; to obtain:

A	a	b	c	d
x^1	1	1	1	0
x^2	0	0	1	1
x^3	1	1	0	1
y^1	1	0	1	0
y^2	0	1	0	1
y^3	0	0	1	1
y^4	0	1	1	0
y^5	1	0	0	1

If the structure is the one of a (functional) game form, the existence of equilibria does not guarantee the *solvability* of the game or the existence of Nash equilibria. *Strong* equilibria are needed. Therefore we will be interested in *strong variations* of Lemma 2. A first one is already implied by the following:

If (w, z) is antagonistic and $(X, Y) \in \mathcal{G}$ is an equilibrium for (w, z), then it is a strong equilibrium; since if $a \in \bigvee(w, Y) \cap \bigvee(z, X)$ and $a' \in X \cap Y$, then $w(a') \leq w(a)$ and $z(a') \leq z(a)$ imply $w(a') = w(a)$ and $z(a') = z(a)$. So, if $\mathcal{G} := \mathcal{H} \times \mathcal{K}$ is a proper and complete structure and $g : \mathcal{G} \to \mathbb{R}$ is a *functionalization* of the structure – i.e., such that $\forall (X, Y) \in \mathcal{H} \times \mathcal{K}$, $g(X, Y) \in X \cap Y$ – then the thus defined (functional) game form $(\mathcal{H}, g, \mathcal{K})$ allows Nash equilibria for all antagonistic games. This, of course, implies the above reminded theorem of [7].

In the next Section 5 we shall drop the restriction to antagonistic games. To close the consideration of such special games, we now present some conclusions that essentially are known since [5] and somehow anticipate our main results of the next Section 5.

Definition. Given a proper structure $\mathcal{G} := \mathcal{H} \times \mathcal{K}$ and a game $(w, z) \in F \times F$,
let $\Lambda(w, \mathcal{K}) := \bigwedge(w, \bigcup\{\bigvee(w, Y); Y \in \mathcal{K}\})$ and $\Lambda(z, \mathcal{H}) := \bigwedge(z, \bigcup\{\bigvee(z, X); X \in \mathcal{H}\})$.

Theorem 1. *Let $\mathcal{G} := \mathcal{H} \times \mathcal{K}$ be a proper structure.*
$\mathcal{G}$ is complete iff for all antagonistic (w, z), any choice $(b, c) \in \Lambda(w, \mathcal{K}) \times \Lambda(z, \mathcal{H})$ yields $w(b) \leq w(c)$.
Then, for all equilibria $(X, Y) \in \mathcal{G}$ of (w, z) and all $a \in X \cap Y$, $w(b) \leq w(a) \leq w(c)$.
$\mathcal{G}$ is coherent iff for all antagonistic (w, z), any choice $(b, c) \in \Lambda(w, \mathcal{K}) \times \Lambda(z, \mathcal{H})$ yields $w(b) \geq w(c)$.
$\mathcal{G}$ is dual iff for all antagonistic (w, z), any choice $(b, c) \in \Lambda(w, \mathcal{K}) \times \Lambda(z, \mathcal{H})$ yields $w(b) = w(c)$.
Then all equilibria $(X, Y) \in \mathcal{G}$ of (w, z), with any $a \in X \cap Y$, yield the same payoffs $(w(a), z(a))$.

Proof. If $\mathcal{G}$ is incomplete, then – Section 2 – there exists a partition $(W, Z) \in \tau(\mathcal{G})$. So the pair (w, z) of characteristic functions of (W, Z) is a competition such that $\forall b \in \Lambda(w, \mathcal{K})$, $w(b) = 1$, and $\forall c \in \Lambda(z, \mathcal{H})$, $z(c) = 1$; so $w(c) = 0$ and thus $w(b) > w(c)$. So, to prove the first assertion, it suffices to reconsider the proof of Lemma 2: If $\mathcal{G}$ is complete, and (w, z) is antagonistic, and $(b, c) \in \Lambda(w, \mathcal{K}) \times \Lambda(z, \mathcal{H})$, then, since $b \in \bigwedge(w, S)$ and $c \in \bigwedge(z, T)$, and since $\exists a \in S \cap T$, we get $w(b) \leq w(a)$, $z(c) \leq z(a)$ and thus $w(a) \leq w(c)$. This also proves the second assertion.

If $\mathcal{G}$ is incoherent, then – Section 2 – there exists a partition $(W, Z) \in \nu(\mathcal{G})$. So the pair (w, z) of characteristic functions of (W, Z) is a competition such that $\forall b \in \Lambda(w, \mathcal{K})$, $w(b) = 0$, and $\forall c \in \Lambda(z, \mathcal{H})$, $z(c) = 0$; so $w(c) = 1$ and thus $w(b) < w(c)$. Instead, if $\mathcal{G}$ is coherent, and (w, z) is antagonistic, and $(b, c) \in \Lambda(w, \mathcal{K}) \times \Lambda(z, \mathcal{H})$, then, if we choose $(X, Y) \in \mathcal{G}$ such that $b \in \bigvee(w, Y)$ and $c \in \bigvee(z, X)$, since $\exists a \in X \cap Y$, $w(b) \geq w(a)$, we get $z(c) \geq z(a)$ and thus $w(b) \geq w(a) \geq w(c)$. This also proves the last two assertions. $\square$

The first part of this Theorem 1 also tells us that completeness of $\mathcal{G}$ is *the* structural property that for any antagonistic conflict encourages a *commitment* of player H (K); i.e., a decision $X \in \mathcal{H}$ $(Y \in \mathcal{K})$, *taken unilaterally and in advance*, that then leaves to player K (H) the *conditioned but final choice* of the situational outcome $a \in X$ $(a \in Y)$. This is so because, if player H *commits* to $X \in \mathcal{H}$, then he *can expect* that player K will choose $c \in \bigvee(z, X)$. Note that all such c do not only have the same $z(c)$, but, since (w, z) is antagonistic, also the same $w(c)$. So X is a *best commitment* for H iff the associated $w(c)$ is *maximal*. Therefore H *can expect* that such a commitment yields an outcome $c \in \bigvee(w, \bigcup\{\bigvee(z, X); X \in \mathcal{H}\}) = \Lambda(z, \mathcal{H})$. This is an outcome that pays him $w(c)$. Iff $\mathcal{G}$ is complete, according to Theorem 1, this amount is at least as much as the payoff $w(b)$ he *could expect* if player K *commits first*.

5. Stability

Besides the *weak* Lemma 2, the results of the last Section 4 only hold for antagonistic games. But these make a very special subclass of situations that do not permit, nor demand, cooperation among the players. To be able to consider general games, we have to focus on properties that are *stronger* than solvability, and therefore need *stronger* hypergraph results:

Definition. Given $z \in F$, we say that $z' \in F$ is a *monotone variation* of z, if $\forall a, b \in A$, $z(a) < z(b)$ implies $z'(a) < z'(b)$. Note that if $z' \in F$ is a monotone variation of z, then $\bigvee(z', B) \subseteq \bigvee(z, B)$ and $\bigwedge(z', B) \subseteq \bigwedge(z, B)$.
Let $\bar{F}$ be the family of injective functions $z' : A \to \mathbb{R}$. Note that $\forall z \in F$, $\exists z' \in \bar{F}$ that is a monotone variation of z: If $a \neq b \in A$ are such that $z(a) = z(b)$, then one can always choose a *sufficiently small* $\epsilon > 0$ and, for instance, redefine $z(b) := z(a) + \epsilon$.
If $z \in \bar{F}$ and $B \subseteq A$ is non-void, then $|\bigvee(z, B)| = 1 = |\bigwedge(z, B)|$; and therefore we shall understand that in this case $\bigvee(z, B)$ ($\bigwedge(z, B)$) denotes the unique *maximal (minimal) element* of B.

Definition. Let $\sigma(z, \mathcal{H}) := \{Z \subseteq A; \forall a \in Z, \exists X \in \mathcal{H}$ such that $X \cap Z = \{a\}$ and $a \in \bigvee(z, X)\}$ denote the family of subsets that are *z-critical, subject to* $\mathcal{H}$. Note that if $Z \in \sigma(z, \mathcal{H})$ and $Y \subseteq Z$, then $Y \in \sigma(z, \mathcal{H})$.
Also note that $\sigma(\mathcal{H}) = \bigcup\{\sigma(z, \mathcal{H}); z \in F\}$; because $\forall z \in F$, $\sigma(z, \mathcal{H}) \subseteq \sigma(\mathcal{H})$, and for any $Z \in \sigma(\mathcal{H})$, if $z \in F$ *respects* Z – because $\forall a \in Z, b \in A \setminus Z, z(a) > z(b)$ – then $Z \in \sigma(z, \mathcal{H})$. So $\sigma(\mathcal{H}) = \bigcup\{\sigma(z, \mathcal{H}); z \in \bar{F}\}$.

Definition. Let $\mathcal{H}$ be a proper hypergraph. Given an injective $z \in \bar{F}$, denote by $\lambda(z, \mathcal{H})$ the *slice, according to z, of* $\mathcal{H}$, i.e., the subset $Z \subseteq A$ that is determined by the following *slicer algorithm*: Start with $Z := \emptyset$; and while $\mathcal{H} \neq \emptyset$, determine $a := \Lambda(z, \mathcal{H}) := \bigwedge(z, \{\bigvee(z, X); X \in \mathcal{H}\})$, redefine $Z := Z \cup \{a\}$ and $\mathcal{H} := \{X \in \mathcal{H}; a \notin X\}$.

Lemma 3. *Let $\mathcal{H}$ be a proper hypergraph. Given $z \in \bar{F}$, $\{\lambda(z, \mathcal{H})\} = \sigma(z, \mathcal{H}) \cap \tau(\mathcal{H})$.*
Therefore $\lambda(\mathcal{H}) = \{\lambda(z, \mathcal{H}); z \in \bar{F}\}$.

Proof. Let us first prove that $\lambda(z, \mathcal{H})$ is a z-critical transversal of $\mathcal{H}$. If $\mathcal{H}$ is void, then $\lambda(z, \mathcal{H}) = \emptyset$ is such a critical transversal. Therefore we may use induction to prove the same for any proper $\mathcal{H}$: Then $a := \bigwedge(z, \{\bigvee(z, X); X \in \mathcal{H}\})$ is well defined, and $\emptyset \notin \mathcal{H}' := \{X \in \mathcal{H}; a \notin X\}$. So we may assume that $Z' := \lambda(z, \mathcal{H}')$ is a z'-critical transversal of $\mathcal{H}'$; i.e., $Z' \in \tau(\mathcal{H}')$ and $\forall a' \in Z', \exists X' \in \mathcal{H}'$ such that $X' \cap Z' = \{a'\}$, $a' = \bigvee(z, X')$ and $a \notin X'$. So, since $\lambda(z, \mathcal{H}) = \{a\} \cup Z'$, it only remains to prove, that for any $X \in \mathcal{H}$ with $a = \bigvee(z, X)$, $X \cap Z' = \emptyset$. To do that, assume $a' \in Z'$. Then, since $\exists X' \in \mathcal{H}'$ with $a' = \bigvee(z, X')$, the choice of a implies $z(a') \geq z(a)$. But since z is injective and $a \notin X'$, $z(a') > z(a)$; and since $a = \bigvee(z, X)$, $a' \notin X$.

Now, to prove that $\lambda(z, \mathcal{H})$ is the only z-critical transversal of $\mathcal{H}$, suppose that $Y, Z \in \tau(\mathcal{H})$ are z-critical but $Y \neq Z$. Clearly $Y \not\subset Z$. So let $a := \bigwedge(z, Y \setminus Z)$, and choose any $X \in \mathcal{H}$ with $\{a\} = X \cap Y$ and $a = \bigvee(z, X)$. Since $Z \in \tau(\mathcal{H})$, there exists $b \in X \cap Z$. Since $b \in Z$, $b \neq a$ and $b \notin Y$; and since $b \in X$, $z(b) < z(a)$. Since Z is z-critical, there exists $X' \in \mathcal{H}$ with $\{b\} = X' \cap Z$ and $b = \bigvee(z, X')$. Since $Y \in \tau(\mathcal{H})$, there exists $a' \in X' \cap Y$. Since $b \notin Y$, $z(a') < z(b)$; so $a' \neq b$, and $a' \notin Z$; but $z(a') < z(a)$; a contradiction. $\square$

Now we finally are prepared to present our main issue and the corresponding result:

Definition. A structure $\mathcal{G} := \mathcal{H} \times \mathcal{K}$ is *stable*, if – it is not only *solvable*, or *stable* in the sense of [9], but – every game $(w, z) \in F \times F$ allows a functional equilibrium.
So, for a structure to be stable, we demand that all potential games can be *functionally solved*. Thus we are adopting a game theoretical perspective that has already motivated some of our early inquiries [13]: We assume that any *unsettled situation* could *destabilize the structure*. Moreover we understand that to *definitely settle* a given situation, a *functional* equilibrium is needed.

Theorem 2. *A proper structure $\mathcal{G} := \mathcal{H} \times \mathcal{K}$ is stable iff it is complete.*

Proof. Lemma 1 already guarantees that if $\mathcal{G}$ is stable, it is complete. So assume that $\mathcal{G}$ is complete; and let $(w, z) \in F \times F$. Let $z' \in \bar{F}$ be a monotone variation of z; and let $Z := \lambda(z', \mathcal{H}) \in \lambda(\mathcal{H})$ – it exists, since $\emptyset \notin \mathcal{H}$. Since $Z \in \tau(\mathcal{H})$, there exists $Y \in \mathcal{K}$ with $Y \subseteq Z$; and thus – Lemma 3 – $Y \in \sigma(z', \mathcal{H}) \subseteq \sigma(z, \mathcal{H})$. Now let $a \in Y$

be such that $a \in \bigvee(w, Y)$ – it exists, since $\emptyset \notin \mathcal{K}$ – and let $X \in \mathcal{H}$ be such that $\{a\} = X \cap Y$ and $a \in \bigvee(z, X)$ – it exists, since $Y \in \sigma(z, \mathcal{H})$. □

This is a fairly simple proof that yields a *structural generalization and functionalization* of an already classical result by Gurvich [8,10]: If the structure of a game form is dual, then all games allow Nash equilibria. But in fact the proof of our Theorem 2 yields a *stronger version* of it, that also suggests a reconsideration of our game theoretical model:

Definition. Given the proper structure $\mathcal{G} := \mathcal{H} \times \mathcal{K}$, we say that $\mathcal{K}$ *is stable in front of* $\mathcal{H}$, if $\forall z \in F$, $\mu(\mathcal{K}) \cap \sigma(z, \mathcal{H}) \neq \emptyset$. The reason is that then, and only then, for any *structurally given* – i.e., invariant, not situation dependent – payoff mapping $z \in F$ of player K, this player can *structurally commit* to a $Y \in \mu(\mathcal{K}) \cap (\sigma(z, \mathcal{H}) \setminus \{\emptyset\})$. This *commitment* Y will be *stable* because it cannot be *questioned by game situations* – now specified by payoff mappings $w \in F$ of player H – since Y can always be *complemented* by a $X \in \mathcal{H}$ to specify a functional equilibrium (Y, X) of (z, w).
Concerning the *solution* of such a game situation $w \in F$ – now predetermined by a *stable commitment* $Y \in \mu(\mathcal{K})$ – the following can be deduced: If player H is now situationally allowed to decide with complete knowledge of Y, then one *can expect* that he will choose a $X \in \mathcal{H}$ such that $X \cap Y$ singles out a unique outcome $a \in Y$ with $a \in \bigvee(w, Y) \cap \bigvee(z, X)$; because such a $X \in \mathcal{H}$ exists, ensures a, given Y, *maximal payoff* for player H, and *discourages reconsideration* of Y by player K. So (X, Y) will be a functional equilibrium of (w, z).

Corollary 1. *If $\mathcal{G} := \mathcal{H} \times \mathcal{K}$ is proper and complete, then $\mathcal{K}$ is stable in front of $\mathcal{H}$; and therefore $\mathcal{G}$ is stable.*

For instance, for the coherent and complete structure we specified in Section 4, consider the following situation (w, z):

A	a	b	c	d
w	3	1	4	0
z	3	4	1	0

The slice $\lambda(z, \mathcal{H})$ we used in the proof of Theorem 2, first only considers z to *commit* player K to $Y^4 \in \mathcal{K}$. Then the *decision* $X^2 \in \mathcal{H}$ and the functional equilibrium (X^2, Y^4) follow. On the other hand, $X^1 = \lambda(w, \mathcal{K})$. So, although (X^2, Y^1) is a functional equilibrium, none of its *determinants* is a *stable commitment*.
Generally it is *harder* to identify *stable commitments* than functional equilibria. But it may be worthwhile in *fuzzy situations*: Since a *stable commitment of one side* only depends on the *structural aspects of the other side*. So *one side's stable commitment will be well defined, even if the other side's payoff function is not.* Therefore we may conclude that structural incompleteness can also turn out to be a handicap in situations that do allow functional equilibria.

6. Proper game structures

Our main result guarantees that, given a game structure, all – an infinite number of – game situations allow solutions, if all – a finite number of – competitions can be dominated by one of the players. But instead of trying to certify that each of the $2^{|A|}$ partitions of A can be responded by the structure, one may want to *prove* stability in polynomial time. Such *succinct certifications* could interest, not only if stability is considered to be a *structural virtue*; but also if it is a *structural hindrance* that should be *uncovered*.

We believe that our result may help to understand certain *remarkable stabilities* of real systems. For example, today's very common *two-major-party systems* seem *antagonistic* at first glance, but they often are *quite stable*. Our game theory would explain this based on the comments that follow Lemma 1, Section 4: If the two major parties manage to *dominate all political elections*, then, resting on the same *solving rationale*, they also will be able to *settle all profitable situations* among them without falling into *situational instabilities* that could be capitalized by third parties.

But, is stability always caused by a *structural reason*? If this would be the case, to be identifiable, the *reason* would have to be *succinctly exposable*; that is in a *time frame* that can polynomially be bounded by the structure's size. Since the *completeness decision problem* is *coNP-complete* [14], so is the *stability decision problem*. Therefore, and since we suppose that *coNP* $\neq$ *NP*, we have to assume that our problem is not in *NP*. So, if we cleave to the until now adopted game theoretical model, we have to conclude that stability does not always allow a polynomial proof. But we feel that this negation of our question is, game theoretically, counterintuitive. This is one of the reasons why we are going to reconsider some of the main notions of our model.

In fact, the adopted notion of *stable commitment* has one *weakness*: If player K commits to a $Y \notin \tau(\mathcal{H})$, then there exist potential decisions $X \in \mathcal{H}$ of player H, with $X \cap Y = \emptyset$; that, if chosen, would not permit any of the, by Y pre-restricted, potential situational outcomes. But such a *no-outcome* is in fact a potential *outcome* that player H may prefer to any other outcome that he could attain; given that player K commits to $Y \in \mu(\mathcal{K})$. Nonetheless, our game-theoretical model does not identify this *no-outcome outcome* as such.

So, if $Y \in \mu(\mathcal{K})$ is to be a *properly stable* commitment of player K – given his payoff function $z \in F$ – it should be a member of $\sigma(z, \mathcal{H}) \cap \tau(\mathcal{H})$. i.e., if $z \in \bar{F}$ – Lemma 3 – $Y = \lambda(z, \mathcal{H})$ has to be the case. So, when $\mathcal{G}$ is proper, $\mathcal{G}$ will be *properly stable*, if $\lambda(\mathcal{G}) \subseteq \mu(\mathcal{G})$. Also note that this *proper completeness* most probably does allow a *succinct certificate*, because – as it easily follows from the results of [6] – the corresponding decision problem is polynomially equivalent to the *duality decision problem*; i.e., is *subexponential* [6], and thus most probably not *NP-hard*.

But we will not dwell on the differences between *completeness* and *proper completeness*, because in the following – for the first time, after Theorem 1 – we shall focus on coherent structures. Since $\mathcal{G} \subseteq \tau(\mathcal{G})$ holds for such structures, $\mathcal{G} \cap \sigma(\mathcal{G}) \subseteq \lambda(\mathcal{G})$ will also. For coherent structures – Section 2 – each of the two considered completeness is equivalent to duality; and for non-void structures, also equivalent to stability.

This is one reason why in the following we shall assume that *proper game structures* are coherent. But the main reason is, of course, that any *proper game structure* must be coherent, because only then it will be a *well-defined game structure*. Otherwise there would exist pairs of potential decisions $(X, Y) \in \mathcal{H} \times \mathcal{K}$ that game theoretically yield *undefined outcomes*.

To illustrate some of the stated problems, consider the very special subclass of *regular structures* $\mathcal{G} := \mathcal{H} \times \mathcal{K}$ for which there exists a pair $h, k \in \mathbb{N}$ with: $\forall X \in \mathcal{H}$, $|X| = h$, and $\forall Y \in \mathcal{K}$, $|Y| = k$. Completeness of $\mathcal{G}$ requires $h + k \leq |A| + 1$; since otherwise there would exist partitions (W, Z) of A with $|W| < h$ and $|Z| < k$. Let us suppose $h + k = |A| + 1$; to guarantee the structure's coherence.

Then the regular structure is complete iff $\mathcal{H} = \{X \subseteq A; |X| = h|\}$ and $\mathcal{K} = \{Y \subseteq A; |Y| = k\}$; because if there would exist a partition (W, Z), with $|W| = h$ but $W \notin \mathcal{H}$, then, since $|Z| = k-1$, the partition would not be responded. Therefore, to prove whether the structure is stable, one can simply *generate* all members of $\{X \subseteq A; |X| = h|\}$ and $\{Y \subseteq A; |Y| = k\}$, and check that they also are in $\mathcal{H}$ and $\mathcal{K}$, respectively. This can, of course, be done in polynomial time.

Definition. Given a game structure $\mathcal{G} := \mathcal{H} \times \mathcal{K}$ we say that it is a *proper game structure*, if it is coherent, non-void and has unique domain.
It is sound to require $\mathcal{H} \neq \emptyset \neq \mathcal{K}$, since we expect that in any situation a pair $(X, Y) \in \mathcal{G}$ will be chosen. So coherence also implies $\emptyset \notin \mathcal{H} \cup \mathcal{K}$.
Note finally that since we require domain uniqueness, in fact we are saving the qualification *proper* for *game structures of proper correspondence game forms*.

Definition. Given any structure $\mathcal{G}$, let the *critical kernel* $\rho(\mathcal{G})$ of $\mathcal{G}$ be the maximal critical substructure of $\mu(\mathcal{G})$. It is well defined since there always exists a unique such substructure; because the structure $\emptyset \times \emptyset$ is critical; and if $\mathcal{G}', \mathcal{G}'' \subseteq \mu(\mathcal{G})$ are critical, then so is $\mathcal{G}' \cup \mathcal{G}''$.

Lemma 4. *Start with $\mathcal{G}' := \mathcal{G}$, and iterate $\mathcal{G}' := \mu(\mathcal{G}') \cap \sigma(\mathcal{G}')$ until convergence. Then, at the end, $\mathcal{G}' = \rho(\mathcal{G})$ will hold. So $\rho(\mathcal{G})$ can be determined in polynomial time.*
If $\mathcal{G}$ is coherent, then $\rho(\mathcal{G})$ is coherent. $\mathcal{G}$ is complete iff $\rho(\mathcal{G})$ is complete. If $\mathcal{G}$ is dual, then $\rho(\mathcal{G}) = \mu(\mathcal{G})$.

Proof. At the end $\mathcal{G}' = \mu(\mathcal{G}') \cap \sigma(\mathcal{G}')$ holds, so $\mathcal{G}' \subseteq \rho(\mathcal{G})$. To prove $\mathcal{G}' \supseteq \rho(\mathcal{G})$, we may use induction: At the beginning, this is clear. So it remains to prove that if $\rho(\mathcal{G}) \subseteq \mathcal{G}'$, then $\rho(\mathcal{G}) \subseteq \mu(\mathcal{G}')$ and $\rho(\mathcal{G}) \subseteq \sigma(\mathcal{G}')$. The first of these inclusions holds, because $\rho(\mathcal{G}) \subseteq \mathcal{G}' \cap \mu(\mathcal{G})$; and since $\mathcal{G}' \subseteq \mathcal{G}$ – see Section 2, Definition of μ – $\mathcal{G}' \cap \mu(\mathcal{G}) \subseteq \mu(\mathcal{G}')$. The second inclusion follows from $\rho(\mathcal{G}) \subseteq \sigma(\rho(\mathcal{G}))$ and the monotony of σ.

If $\mathcal{G}$ is dual, then $\mu(\mathcal{G}) = \lambda(\mathcal{G})$ implies $\mu(\mathcal{G}) \subseteq \sigma(\mathcal{G})$, and therefore $\rho(\mathcal{G}) = \mu(\mathcal{G})$. Since $\rho(\mathcal{G}) \subseteq \mathcal{G}$, if $\mathcal{G}$ is coherent, so is $\rho(\mathcal{G})$. Moreover, if $\rho(\mathcal{G})$ is complete, so is $\mathcal{G}$. So it only remains to prove that if $\mathcal{G}$ is complete, so is $\rho(\mathcal{G})$. Assume that $\lambda(\mathcal{G}) \subseteq \nu(\mathcal{G})$ and define $\mathcal{G}' := \mu(\mathcal{G}) \cap \sigma(\mathcal{G})$. Since $\lambda(\mathcal{G}) \subseteq \sigma(\mathcal{G})$ and – see Section 2, Definition of σ – $\nu(\mathcal{G}) \cap \sigma(\mathcal{G}) \subseteq \nu(\mathcal{G}')$, we get $\lambda(\mathcal{G}) \subseteq \nu(\mathcal{G}')$. So, *applying* τ on both sides, yields $\tau(\mathcal{G}') \subseteq \nu(\mathcal{G})$, and therefore

$\lambda(\mathcal{G}') \subseteq \nu(\mathcal{G})$. Since $\lambda(\mathcal{G}') \subseteq \sigma(\mathcal{G}') \subseteq \sigma(\mathcal{G})$, we get $\lambda(\mathcal{G}') \subseteq \nu(\mathcal{G}) \cap \sigma(\mathcal{G}) \subseteq \nu(\mathcal{G}')$. Iterating this result, yields $\lambda(\rho(\mathcal{G})) \subseteq \nu(\rho(\mathcal{G}))$. $\square$

The stability decision problem of proper structures can thus be polynomially reduced to the same problem, but restricted to minimal and critical structures. If the original structure is coherent, then the reduced one will be a proper game structure. So, if one *functionalizes* the correspondence game form of such a proper game structure, simply choosing always one of the existing allowed outcomes, then this functional game form will be *solvable* iff the structure is complete.

The structures of our first example – Sections 3 and 4 – are critical, proper game structures. One *functionalization* of the *completed structure* – Section 4 – gives raise to the following solvable *extension* of the original – Section 3 – game form of our example:

g	y_1	y_2	y_3	y_4	y_5
x_1	a	b	c	b	a
x_2	c	d	d	c	d
x_3	a	b	d	b	a

This proves that if one focuses on stability of coherent structures, then one may as well restrict the attention to (functional) game forms. This is a result that we only have seen mentioned in [11]:

Corollary 2. *The stability decision problem, restricted to coherent structures – or the duality decision problem of general structures – can polynomially be reduced to the* solvability problem of (functional) game forms.

But this solvability problem, as the duality problem, has not yet been proven to be polynomial. In fact it has not even been proven to be a member of *NP*. Does duality have *succinct reasons* to be? We do not have an answer. We only know that the question is important; and we hope that our results will not only contribute to emphasize the relevance of this question, but also to its eventual answer.

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